SEMICLASSICAL ANALYSIS FOR MATRIX-VALUED OPERATORS AND APPLICATIONS TO SPECTRAL ASYMOTOTICS

SETSURO FUJIIE

Abstract. This short manuscript is made for the mini-course in the RIMS workshop「非コンパクト空間上のシュレディンガー作用素の半古典解析とスペクトル理論」held in RIMS, Kyoto University, on September 3,4 and 5, 2025.

When a semiclassical differential operator is matrix-valued, at least two interesting problems arise. First, the eigenvalues of the principal symbol (assumed Hermitian) may have singularities as functions in the phase space at crossing points where their multiplicity changes. Second, even if the the principal symbol is regularly diagonalized, the so-called *non-adiabatic transition* occurs as was first suggested by Landau and Zener for a simple model. This model implies that if two classical trajectories cross transversally at a point, the transition probability from one to the other is of order $h^{1/2}$, where h is the semiclassical parameter.

In this mini-course, we focus on this seconde problem, and study a model of a 1D 2×2 matrix Schrödinger operator (1.1), where the principal part is diagonal with two Schrödinger operators. The goal is to understand that the transition, which we express by the off-diagonal entries of the *microlocal scattering matrix*, is of order $h^{\frac{1}{m+1}}$ when the contact order of the crossing is m. We apply this microlocal result to the semiclassical asymptotic distribution of eigenvalues and resonances.

Contents

1. Introduction and survey

In this minicourse, we mainly consider the matrix Schrödinger operator in dimension 1:

$$\mathscr{P} = \begin{pmatrix} P_1 & hW \\ hW & P_2 \end{pmatrix}, \tag{1.1}$$

where each P_j $(j = 1, 2)$ is a scalar Schrödinger operator

$$P_j = -h^2 \frac{d^2}{dx^2} + V_j(x),$$

with a real-valued smooth potential $V_j(x)$, and W is a multiplication operator by a real-valued smooth function $W(x)$. The constant h is regarded as a positive small parameter (semiclassical parameter). Such a matrix-valued operator $\mathscr{P}$ appears in the quantum chemistry as the Born-Oppenheimer approximation (see [24]), where the semiclassical parameter corresponds to the square root of the ratio of the mass of the nuclear and the electron.

Let $p_j(x, \xi)$ be the classical Hamiltonian corresponding to the Schrödinger operator P_j;

$$p_j(x, \xi) = \xi^2 + V_j(x),$$

and H_{p_j} the Hamiltotnian vector field;

$$H_{p_j} := \frac{\partial p_j}{\partial \xi}\frac{\partial}{\partial x} - \frac{\partial p_j}{\partial x}\frac{\partial}{\partial \xi} = 2\xi \frac{\partial}{\partial x} - V_j'(x)\frac{\partial}{\partial \xi}.$$

The value $p_j(x, \xi)$ is invariant along the integral curve $\exp tH_{p_j}(x_0, \xi_0)$ (energy conservation).

1.1. **Basic facts for the scalar Schrödinger operators.** Let us review some basic facts for the scalar Schrödinger operator $P = -h^2 \frac{d^2}{dx^2} + V(x)$. We denote $p(x, \xi) = \xi^2 + V(x)$. We consider the following conditions on the potential $V(x)$ and the energy level $E_0 \in \mathbb{R}$ near which we study the eigenvalues and resonances.

Condition 1. The function $V(x)$ is real-valued on $\mathbb{R}$ and analytic in an angular complex neighborhood of $\mathbb{R}$

$$\mathscr{S} := \{x \in \mathbb{C}; |\mathrm{Im} x| < (\tan \theta_0)(1 + |\mathrm{Re} x|)\}$$

for some positive constant $\theta_0 < \pi/2$. Moreover, $V(x)$ have limits different from E_0 as $\mathrm{Re} x \to \pm\infty$ in this domain;

$$V(x) \to V^{\pm} \neq E_0 \quad \text{as } \mathrm{Re} x \to \pm\infty \text{ in } \mathscr{S}. \tag{1.2}$$

Condition 2$_\pm$. There exists $c_0 \in \mathbb{R}$ such that

$$\pm \frac{V(x) - E_0}{x - c_0} > 0 \quad \forall x \in \mathbb{R}.$$

Condition 3. There exist $a_0 < b_0$ such that

$$\frac{V(x) - E_0}{(x - a_0)(x - b_0)} > 0 \quad \forall x \in \mathbb{R}.$$

Under the condition $2_\pm$, the *energy surface* (or *characteristic set*)

$$\Gamma(E) = \{(x, \xi); p(x, \xi) = E\}$$

is an unbounded curve for E close enough to E_0. The classical trajectory for p starting from a point in $\Gamma(E)$ goes to infinity as t tends to plus and minus infinity under the condition 1.

The spectrum of the scalar operator P is continuous near E_0. Moreover, it is known that there is no resonance in a small complex neighborhood of E_0.

Fact 1. ([30]) *Suppose that $E_0 \in \mathbb{R}$ is a non-trapping energy, i.e.*

$$|\exp tH_p(x_0,\xi_0)| \to \infty \text{ as } t \to \pm\infty \text{ for any } (x_0,\xi_0) \in \Gamma(E) = p^{-1}(E).$$

Then there is no resonance in any complex neighborhood of size $\mathcal{O}(h|\log h|)$ of E_0.

Under the condition 3, on the other hand, $\Gamma(E)$ is a simple closed curve for E close enough to E_0. The classical trajectory for p starting from a point in $\Gamma(E)$ is periodic along $\Gamma(E)$. The spectrum of P near E_0 consists of (simple) eigenvalues. Let $\mathcal{A}(E)$ be the action of the classical Hamiltonian $p(x,\xi)$ defined as line integral of the one form ξdx along this curve:

$$\mathcal{A}(E) := \int_{\Gamma(E)} \xi dx,$$

which is the volume of the domain bounded by the closed curve $\Gamma(E)$. It is a smooth function of E with $\mathcal{A}'(E) > 0$, and the derivative $\mathcal{A}'(E)$ is the period of the classical trajectory on $\Gamma(E)$.

Fact 2. *In the semiclassical limit $h \to +0$, they are approximated by E's satisfying the so-called Bohr-Sommerfeld quantization rule (see [21], [32], [35]):*

$$-e^{i\mathcal{A}(E)/h} = 1. \tag{1.3}$$

The condition (1.3) can be rewritten as

$$\mathcal{A}(E) = (2k+1)\pi h, \quad k \in \mathbb{Z},$$

and hence the eigenvalues near E_0 is a sequence with interval $\sim 2\pi h/\mathcal{A}'(E_0)$.

1.2. Three models of matrix Schrödinger operators.

Now we come back to the matrix-valued operator $\mathscr{P}$ defined by (1.1).

Let $\Gamma_j(E)$ be the characteristic set of each P_j;

$$\Gamma_j(E) := \{(x,\xi) \in \mathbb{R}^2; p_j(x,\xi) = E\} = p_j^{-1}(E). \tag{1.4}$$

We are interested in the case $\Gamma_1(E) \cap \Gamma_2(E) \neq \emptyset$. We call *crossing points* the elements of $\Gamma_1(E) \cap \Gamma_2(E)$. If (x,ξ) is a crossing point, then $V_1(x) = V_2(x) =: V_c$ and $\xi^2 = E - V_c$, and hence $V_c \leq E$, namely the crossing value is below or at the energy level.

Here we always assume

Assumption 1. *The potentials $V_1(x)$ and $V_2(x)$ satisfy the condition 1, and $W(x)$ is a real-valued smooth function on $\mathbb{R}$ extended to a bounded analytic function in $\mathscr{S}$.*

Assumption 2. *The potentials cross at one point $x = 0$ below E_0, more precisely,*

$$\{x \in \mathbb{R}; V_1(x) = V_2(x) \leq E_0\} = \{0\}, \quad V_1(0) = V_2(0) = 0 < E_0, \tag{1.5}$$

and the contact order is finite: for $m = 1, 2, \ldots,$ one has

$$V_1^{(k)}(0) - V_2^{(k)}(0) = 0 \quad (0 \leq k \leq m-1), \qquad V_1^{(m)}(0) - V_2^{(m)}(0) \neq 0. \tag{1.6}$$

Assumption 2 implies that $\Gamma_1(E_0)$ and $\Gamma_2(E_0)$ intersect at two points $(0,\sqrt{E_0})$ and $(0,-\sqrt{E_0})$ and the contact order at these points are both m. In the case where $E_0=0$, the crossing point $x=0$ is also a turning point of each V_j. $\Gamma_1(0)$ and $\Gamma_2(0)$ intersect at one point $(0,0)$ with contact order $2m$. This case was studied in [4], but we avoid it here for the simplicity of the presentation.

1.2.1. **Model A: non-trapping $\times$ non-trapping.** Let us first consider the case where E_0 is non-trapping for both V_1 and V_2. In addition to Assumptions 1 and 2, we assume condition 2_+ for V_1 with turning point c_1 and condition 2_- for V_2 with c_2 such that $c_2<c_1$. Then there exists a *directed cycle* composed by the classical trajectories of p_1 and p_2. Let $S(E)$ be the volume of the domain bounded by the directed cycle. The following theorem by K. Higuchi says that a directed cycle may produce resonances near the non-trapping energy for both P_1 and P_2.

Theorem 1.1. ([17],[18]) *Assume $m=1$. For $h>0$ small enough, there exist resonances E near E_0 such that*

$$\operatorname{Im} E \sim -\frac{1}{S'(E_0)} h \log\frac{1}{h}.$$

Remark 1.1. *In the case where both V_1 and V_2 satisfy condition 2_+ (or 2_-), we can show that there is no resonance with imaginary part of order $h\log\frac{1}{h}$.*

1.2.2. **Model B: non-trapping $\times$ periodic.** Suppose now that V_1 satisfies the condition 3 while V_2 satisfies the condition 2. P_1 has eigenvalues near E_0 approximated for small h by the energies E satisfying the Bohr-Sommerfeld rule (1.3) with the action integral $\mathcal{A}$ for the potential V_1. The following theorem says that the eigenvalues of P_1 shift in the complex plane transforming into resonances for $\mathscr{P}$. The imaginary part of resonances is of polynomial order of h and the order is $\frac{m+3}{m+1}=1+\frac{2}{m+1}$. This implies that the quantum particle can escape more easily from the trapped trajectory when the contact order is larger.

Theorem 1.2. ([12] ($m=1$), [3]) *For each small h and $\lambda=\lambda(h)\in\mathbb{R}$ near E_0 satisfying the BS rule, there exists a resonance E with $|E-\lambda|=\mathcal{O}(h^2)$ such that*

$$\operatorname{Im} E \sim -D(\lambda) h^{\frac{m+3}{m+1}},$$

where $D(\lambda)$ is a constant independent of h explicitly computable. For example in the case $m=1$, one has

$$D(\lambda)=\frac{2\pi W(0)^2}{\sqrt{\lambda}\mathcal{A}'(\lambda)(V_2'(0)-V_1'(0))}\sin^2\left(\frac{S(\lambda)}{h}-\frac{\pi}{4}\right).$$

Here $S(\lambda)$ is the volume of the domain bounded by the directed cycle as in the previous theorem.

Remark 1.2. *When the interaction W vanishes at the crossing point $x=0$, the top term coefficient $D(\lambda)$ vanishes too. V. Louatron has studied such a case in [27] and computed the first term of the imaginary part of resonances. He showed that the leading order in h becomes $\frac{m+3+2k}{m+1}$ when the vanishing order of W at $x=0$ is k.*

1.2.3. **Model C: periodic $\times$ periodic.** Finally we suppose that both V_1 and V_2 satisfy the condition 3, in addition to Assumptions 1 and 2. This implies that the two turning points a_j,b_j of each V_j satisfy for example $a_1<a_2<b_1<b_2$. It is known that the spectrum of $\mathscr{P}$ near E_0 consists of eigenvalues, and they are approximated by the union of the eigenvalues of P_1 and the ones of P_2.

In order to measure the interaction between the two well $[a_1, b_1]$ and $[a_2, b_2]$, we study the symmetric case $V_1(x) = V_2(-x)$, where the operators P_1 et P_2 have exactly the same eigenvalues. Due to the interaction between the two wells, the so-called *eigenvalue splitting* occurs and a couple of eigenvalues E_+, E_- corresponding to a same eigenvalue λ of P_1, P_2 have a small distance concerning the strength of the interaction. In the well-known case of a scalar Schrödinger operator with a symmetric double well potential, the interaction is caused by the tunneling effect through the barrier between the wells, and the splitting is exponentially small in h. The exponential rate is the Agmon distance between the two wells. In the present case, the interaction is governed by the change of trajectory at crossing points an the splitting is of polynomial order in h.

Theorem 1.3. ([2]) *Assume $m = 1$. For each small h and $\lambda = \lambda(h)$ near E_0 satisfying the BS rule for $P_1 = P_2$, there exist two eigenvalues E^+ et E^- of $\mathscr{P}$ with $|E^\pm - \lambda| = \mathcal{O}(h^{3/2})$ such that*

$$|E^+ - E^-| \sim D(\lambda)h^{3/2},$$

where the constant is given, with $v := V_1'(0) = -V_2'(0)$, by

$$D(\lambda) = \sqrt{\frac{2\pi}{v}}\frac{|W(0)|}{\lambda^{1/4}A'(\lambda)}\left|\cos\left(\frac{S(\lambda)}{h} + \frac{\pi}{4}\right)\right|.$$

1.3. **Microlocal method.** In the next section, we introduce the space of microlocal solutions $\mathscr{E}_{(x_0,\xi_0)}(\mathscr{P} - E)$ to the system $(\mathscr{P} - E)\mathbf{u} = 0$ at each point (x_0, ξ_0) of the phase space $\mathbb{R}_x \times \mathbb{R}_\xi$: $\mathbf{u} \in \mathscr{E}_{(x_0,\xi_0)}(\mathscr{P} - E)$ if $(\mathscr{P} - E)\mathbf{u} \equiv 0$ at (x_0, ξ_0) in the sense of Definition 2.1.

As in the three models, we assume that P_j is of real principal type for each $j = 1, 2$, namely, $dp_j \neq 0$ if $p_j = 0$. We will see the following facts:

Proposition 1.3. *Let $\Gamma_c(E) = \Gamma_1(E) \cap \Gamma_2(E)$ be the set of crossing points. Then one has*

$$\dim \mathscr{E}_{(x_0,\xi_0)}(\mathscr{P} - E) = \begin{cases} 0 & \textit{if } (x_0, \xi_0) \notin \Gamma_1(E) \cup \Gamma_2(E), \\ 1 & \textit{if } (x_0, \xi_0) \in (\Gamma_1(E) \cup \Gamma_2(E)) \setminus \Gamma_c(E), \\ 2 & \textit{if } (x_0, \xi_0) \in \Gamma_c(E). \end{cases}$$

Proof. The first two statements are analogous to the scalar case. The first one is due to the microlocal ellipticity of $\mathscr{P} - E$ near (x_0, ξ_0), namely $\det(\mathscr{P}(x_0, \xi_0) - E) \neq 0$. If (x_0, ξ_0) is on $\Gamma_1(E) \setminus \Gamma_2(E)$, for example, the problem is reduced to the scalar case for $P_1 - E$, and it is well known that the operator is microlocally reduced to the operator hD_x (see for example the text by Zworski [36]), which implies that $\dim \mathscr{E}_{(x_0,\xi_0)}(\mathscr{P} - E) = 1$ in our one dimensional setting. The space $\mathscr{E}_{(x_0,\xi_0)}(\mathscr{P} - E)$ is generated by a Lagrangian distribution associated with the classical trajectory γ which (x_0, ξ_0) belongs to. We will denote this space by $\mathscr{E}_\gamma(\mathscr{P} - E)$.

The last statement will be proved using Theorem 2.1 of the next section, at least in the generic case where the crossing point is not a turning point. In such a case, the crossing point is *microhyperbolic* in the direction $(\operatorname{sign}\xi_0, 0)$ in the sense of Definition 2.5. Let $\gamma_{1,\flat} \subset \Gamma_1(E)$ and $\gamma_{2,\flat} \subset \Gamma_2(E)$ be the two incoming classical trajectories (with the orientation by the time evolution) to the crossing point (x_0, ξ_0), and $\gamma_{1,\sharp} \subset \Gamma_1(E)$ and $\gamma_{2,\sharp} \subset \Gamma_2(E)$ the outgoing ones. Then Theorem 2.1 implies that if a microlocal solution u at the crossing point (x_0, ξ_0) is microlocally zero both on $\gamma_{1,\flat}$ and on $\gamma_{2,\flat}$, then it is microlocally zero at (x_0, ξ_0) (and hence both on $\gamma_{1,\sharp}$ and on $\gamma_{2,\sharp}$). This implies that $\dim \mathscr{E}_{(x_0,\xi_0)}(\mathscr{P} - E) \leq 2$.

On the other hand, one can construct WKB solutions $\mathbf{f}_{\gamma_j}$, $j=1,2$, such that $\mathbf{f}_{\gamma_j}$ generates $\mathscr{E}_{\gamma_j}(\mathscr{P}-E)$ and that $\mathbf{f}_{\gamma_j}\equiv 0$ on γ_k for $k\neq j$. This implies that $\dim \mathscr{E}_{(x_0,\xi_0)}(\mathscr{P}-E)\geq 2$, and the third statement of the proposition is proved. □

The previous proposition leads to the definition of what we call *microlocal scattering matrix* at each crossing point $(x_0,\xi_0)\in\Gamma_1(E)\cap\Gamma_2(E)$. As we saw in the proof, the microlocal data on the incoming classical trajectories $\gamma_{1,\flat}$ and on $\gamma_{2,\flat}$ determine those of the outgoing classical trajectories $\gamma_{1,\sharp}$ and on $\gamma_{2,\sharp}$.

Definition 1.4. There exists an h-dependent 2×2 constant matrix T such that if $\mathbf{u}\in\mathscr{E}_{(x_0,\xi_0)}(\mathscr{P}-E)$, and if

$$\mathbf{u}\equiv\alpha_{j,\flat}\mathbf{f}_{j,\flat}\quad\text{on } \gamma_{j,\flat},$$
$$\mathbf{u}\equiv\alpha_{j,\sharp}\mathbf{f}_{j,\sharp}\quad\text{on } \gamma_{j,\sharp},$$

then

$$\begin{pmatrix}\alpha_{1,\sharp}\\ \alpha_{2,\sharp}\end{pmatrix}=T\begin{pmatrix}\alpha_{1,\flat}\\ \alpha_{2,\flat}\end{pmatrix}. \tag{1.7}$$

Remark 1.5. *The definition of T depends on the choice of the generators $\mathbf{f}_{j,\flat}$ and $\mathbf{f}_{j,\sharp}$.*

The main goal of this course is the following semiclassical asymptotic formula of the microlocal scattering matrix (which will be restated as Theorem 4.1), which says that, for a suitable choice of the WKB solutions $\mathbf{f}_{\gamma_j}$, $j=1,2$ (such that the phase function vanishes at the crossing point), it is an identity matrix at the principal level and the second term is off-diagonal of order $h^{\frac{1}{m+1}}$. This subprincipal term, which stands for the change of trajectory by the quantum particles, concerns the imaginary part of resonances or the splitting of eigenvalues.

Theorem 1.4. *There exists a constant ω (given in Theorem 4.1) such that one has*

$$T=\mathrm{Id}-ih^{\frac{1}{m+1}}\begin{pmatrix}0&\omega\\ \overline{\omega}&0\end{pmatrix}+\mathcal{O}(h^{\frac{2}{m+1}})\quad \text{as } h\to 0. \tag{1.8}$$

The union $\Gamma_1(E)\cup\Gamma_2(E)$ can be regarded as finite directed graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$ with the set of vertices $\mathcal{V}$ and the set of edges $\mathcal{E}$ when the crossing points and classical trajectories are regarded as vertices and edges respectively.

On a graph $\mathcal{G}=(\mathcal{V},\mathcal{E})$, we define a *monodromy matrix* $M=M(E,h)$.

Definition 1.6. M is the matrix the size of which is the number of finite edges $\sharp\mathcal{E}_{\mathrm{fin}}$:

$$M=(m_{e,e'})_{e,e'\in\mathcal{E}_{\mathrm{fin}}}.$$

The entries are defined by

$$m_{e,e'}=\begin{cases}(T_v)_{jk} & \text{if } e^-=(e')^+=v,\ e\subset\Gamma_j, e'\subset\Gamma_k,\\ 0 & \text{if } e^-\neq(e')^+,\end{cases}$$

where $(T_v)_{jk}$ is the (j,k) entry of the microlocal scattering matrix T_v at the vertice v, and e^-, e^+ stand for the starting point and the endpoint respectively of the edge e.

Remark 1.7. *The monodromy matrix M is independent of the choice of $\mathbf{f}_{j,\flat}$ and $\mathbf{f}_{j,\sharp}$.*

Let us illustrate the situation in the case of model B.

There are two vertices $v_+ = (0, \sqrt{E})$ and $v_- = (0, -\sqrt{E})$ and 5 edges $e_1, e_2 \subset \Gamma_1(E)$, $e_3, e_4, e_5 \subset \Gamma_2(E)$ such that $e_1^+ = v_+$, $e_2^+ = v_-$ and $e_3^- = v_+$, $e_3^+ = v_-$, $e_4^+ = v_+$, $e_5^- = v_-$. The edges e_4 and e_5 are not finite and there are only three finite edges e_1, e_2, e_3. Then the monodromy matrix M is of the form

$$M = \begin{pmatrix} 0 & (T_{v_-})_{11} & (T_{v_-})_{12} \\ (T_{v_+})_{11} & 0 & 0 \\ (T_{v_+})_{21} & 0 & 0 \end{pmatrix}$$

Proposition 1.8. *The eigenvalues or the resonances near E_0 are approximated by the set of E's satisfying the condition (see [2], [3] for the precise description)*

$$\det(M(E,h) - I) = 0. \tag{1.9}$$

In the case of Model B, the condition (1.9) gives

$$1 - (T_{v_-})_{11}(T_{v_+})_{11} - (T_{v_-})_{12}(T_{v_+})_{21} = 0. \tag{1.10}$$

Regardless of the choice of the WKB solutions on the edges, the off-diagonal entries $(T_{v_-})_{12}$ and $(T_{v_+})_{21}$ are of order $h^{\frac{1}{m+1}}$. Hence the first approximation that the condition (1.10) supplies us is

$$1 - (T_{v_-})_{11}(T_{v_+})_{11} = \mathcal{O}(h^{\frac{2}{m+1}}).$$

The diagonal entry $(T_{v_+})_{11}$ is 1 modulo $\mathcal{O}(h^{\frac{1}{m+1}+\epsilon})$ if the WKB solutions on e_1 and on e_2 are chosen with phase base point at v_+. If the other diagonal entry $(T_{v_-})_{11}$ is computed with these WKB solutions, it becomes $\exp i\left(\mathcal{A}(E)/h - \pi\right)$ modulo $\mathcal{O}(h^{\frac{1}{m+1}+\epsilon})$. π is the Maslov correction arising from the turning points $(a(E), 0)$ and $(b(E), 0)$.

In order to obtain the asymptotics of the imaginary part of resonances, we should take the smaller term $(T_{v_-})_{12}(T_{v_+})_{21}$ into account. But for this purpose, Theorem 4.1 is not sufficient. We also need the second terms of the diagonal entries which are in the error in this theorem.

There is an alternative way to compute the imaginary part of resonances. Let $\hat{x}$ be a point near $-\infty$. It is in the classically allowed region for P_2. We compute the scalar product in the space $L^2([\hat{x}, +\infty))$ of a resonant state $\mathbf{w}$ with $(\mathscr{P} - E)\mathbf{w}$, which is zero, to have

$$\begin{aligned} 0 &= \langle (\mathscr{P} - E)\mathbf{w}, \mathbf{w}\rangle_{L^2([\hat{x},+\infty))} \\ &= \|hD_x\mathbf{w}\|^2_{L^2([\hat{x},+\infty))} - E\|\mathbf{w}\|^2_{L^2([\hat{x},+\infty))} - h^2\langle \mathbf{w}'(\hat{x}), \mathbf{w}(\hat{x})\rangle. \end{aligned}$$

Taking the imaginary part of this identity, we obtain an expression of the imaginary part of the resonance in terms of the values of the normalized resonant state $\mathbf{w}$ and its derivative at the point $\hat{x}$: If we write $\mathbf{w} = {}^t(w_1, w_2)$, we have

$$\operatorname{Im} E \cdot \|\mathbf{w}\|^2_{L^2([\hat{x},+\infty))} = -h^2(w_1'(\hat{x})\overline{w_1(\hat{x})} + w_2'(\hat{x})\overline{w_2(\hat{x})}). \tag{1.11}$$

Now we look at this identity from the microlocal point of view. Suppose

$$\mathbf{w} \equiv \alpha_j \mathbf{f}_{e_j} \quad \text{on } e_j, \quad j = 1, \dots 5.$$

We normalize $\mathbf{w}$ such that $\alpha_1 = 1$ for example. The fact that $\mathbf{w}$ is a resonant state implies that it is microlocally zero on the incoming trajectory e_4, i.e. $\alpha_4 = 0$. Then we see from the microlocal scattering matrix at the crossing points that $\mathbf{w}$ is microlocally supported only on $\Gamma_1(E)$ at the principal level, and it follows that

$$\|\mathbf{w}\|^2_{L^2([\hat{x},\infty)} \sim 2\mathcal{A}(E).$$

On the other hand, the right hand side of (1.11) is governed by the second term $w_2'(\hat{x})\overline{w_2(\hat{x})}$, because $\hat{x}$ is outside the x-space projection of $\Gamma_1(E)$. Furthermore, $\mathbf{w}$ is microlocally zero on the incoming trajectory e_4, and hence the right hand side of (1.11) is written in terms of α_5. Writing w_2 and w_2' in the WKB form, we obtain the following formula from which we can compute the asymptotics of the imaginary part of resonances.

Lemma 1.9. *Let E a resonance near E_0. Then one has*

$$\operatorname{Im} E = -\frac{h|\alpha_5|^2}{2\mathcal{A}'(E)} + \mathcal{O}(h^{\frac{m+3}{m+1}+\delta}).$$

2. Basic microlocal study

2.1. **Propagation of singularities.** Let $H(x,\xi) = (H_{jk}(x,\xi))_{j,k=1}^N$ be a $N \times N$ matrix-valued smooth function on the phase space $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$ satisfying

$$|\partial_x^\alpha \partial_\xi^\beta H_{j,k}(x,\xi)| \leq C_{\alpha,\beta}$$

for all $1 \leq j,k \leq N$. We call *symbol* such a matrix-valued function in $S(1)$. To each $H(x,\xi) \in S(1)$, we associated an operator

$$H^W(x,hD)u := \frac{1}{(2\pi h)^n} \iint e^{\frac{i}{h}(x-y)\cdot\xi} H\left(\frac{x+y}{2},\xi\right) u(y)dyd\xi$$

applied to a vector-valued function $u = {}^t(u_1,\dots,u_N) \in \mathscr{S}(\mathbb{R}^n,\mathbb{C}^N)$. We call this operator *Weyl quantization* of the symbol $H(x,\xi)$. It is extended to a bounded operator in $L^2(\mathbb{R}^n,\mathbb{C}^N)$.

Definition 2.1. Let (x_0,ξ_0) be a point in $\mathbb{R}^n_x \times \mathbb{R}^n_\xi$, and $u(x,h) = {}^t(u_1,\dots,u_N)$ be a vector-valued function in $L^2(\mathbb{R}^n,\mathbb{C}^N)$ with $\|u\|_{L^2(\mathbb{R}^n,\mathbb{C}^N)} \leq 1$. We say that u is *microlocally infinitely small* (or more simply *microlocally zero*) at (x_0,ξ_0) if there exists $\chi(x,\xi) \in S(1)$ such that $\chi(x_0,\xi_0) = 1$ and

$$\|\chi^W(x,hD)u\| = \mathcal{O}(h^\infty).$$

The complementary set of such points is called *semiclassical wave front set* and denoted $\mathrm{WF}_h(u)$. We say that u is a microlocal solution to the system $H^W u = 0$ at (x_0,ξ_0) if $H^W u \equiv 0$ at (x_0,ξ_0). We denote $\mathscr{E}_{(x_0,\xi_0)}(H^W)$ the vector space of microlocal solutions to the system $H^W u = 0$ at (x_0,ξ_0).

Proposition 2.2. *For a function $u(x,h)$ of the WKB form*

$$u(x,h) = e^{\frac{i}{h}\phi(x)}a(x),$$

where $\phi \in C^\infty(\mathbb{R}^n,\mathbb{R})$ is a real and $a(x) \in C_0^\infty(\mathbb{R}^n,\mathbb{C}^N)$, we have

$$\mathrm{WF}_h(u) = \left\{\left(x, \frac{\partial\phi}{\partial x}(x)\right); x \in \operatorname{supp} a\right\}.$$

Now we allow that the symbol H depends on h:

$$H(x,\xi;h) = H_0(x,\xi) + hR(x,\xi;h), \quad H_0, R \in S(1),$$

and assume that H_0, the principal symbol, is Hermitian.

Example. *The operator $\mathscr{P}$ is the Weyl quantization of the symbol*

$$\begin{pmatrix} \xi^2 + V_1(x) & hW(x) \\ hW(x) & \xi^2 + V_2(x) \end{pmatrix} = \begin{pmatrix} \xi^2 + V_1(x) & 0 \\ 0 & \xi^2 + V_2(x) \end{pmatrix} + h \begin{pmatrix} 0 & W(x) \\ W(x) & 0 \end{pmatrix}.$$

Remark that the principal symbol is a diagonal matrix.

Let $u(x,h) \in L^2(\mathbb{R}^n;\mathbb{C}^N)$ satisfying $\|u\| \le 1$ be a solution to the system

$$H^W(x,hD)u = 0. \tag{2.1}$$

We first have the following microlocal property of u.

Proposition 2.3. *The semiclassical wave front set is included in the characteristic set:*

$$\mathrm{WF}_h(u) \subset \Gamma := \{(x,\xi); \det H_0(x,\xi) = 0\}.$$

Remark 2.4. *If we denote $\lambda_1(x,\xi) \le \lambda_2(x,\xi) \le \cdots \le \lambda_N(x,\xi)$ the eigenvalues of the matrix $H_0(x,\xi)$, the characteristic set is expressed by*

$$\Gamma = \bigcup_{j=1}^{N} \{(x,\xi); \lambda_k(x,\xi) = 0\}.$$

In the case $H^W = \mathscr{P} - E$, we have $\Gamma = \Gamma_1(E) \cup \Gamma_2(E)$, where $\Gamma_j(E)$ is defined by (1.4).

We now state a propagation of singularity theorem in the case of system. In this case, the real principal type condition is generalized to the microhyperbolicity condition in the sense of Ivrii [22].

Definition 2.5. A Hermitian symbol $H_0(x,\xi)$ is said to be *microhyperbolic* at a point $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ in the direction $(x^*,\xi^*) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus (0,0)$ if there exists a constant $C > 0$ such that for any $\omega \in \mathbb{C}^N$ we have

$$\langle \partial_{(x^*,\xi^*)} H_0(x_0,\xi_0)\omega, \omega\rangle \ge \frac{1}{C}\|\omega\|^2 - C\|H_0(x_0,\xi_0)\omega\|^2. \tag{2.2}$$

where $\partial_{(x^*,\xi^*)} := x^* \cdot \partial_x + \xi^* \cdot \partial_\xi$ is the directional derivative in the direction (x^*,ξ^*).

At any point (x_0,ξ_0) satisfying $\det H_0(x_0,\xi_0) \ne 0$, $H_0(x,\xi)$ is microhyperbolic in any direction $(x^*,\xi^*) \in \mathbb{R}^{2n} \setminus (0,0)$. In the scalar case $N = 1$, $H_0(x,\xi)$ is microhyperbolic at a point (x_0,ξ_0) satisfying $H_0(x_0,\xi_0) = 0$ in some direction if and only if $\nabla H_0(x_0,\xi_0) \ne 0$.

Now let us consider a simple example of operator of the form $\mathscr{P}$ whose principal symbol is

$$\begin{pmatrix} \xi^2 + a_1 x & 0 \\ 0 & \xi^2 + a_2 x \end{pmatrix}, \quad a_1 a_2 \ne 0,\ a_1 < a_2.$$

The characteristic sets $\Gamma_j = \{(x,\xi) \in \mathbb{R}^2; \xi^2 + a_j x = 0\}$, $j = 1,2$ are parabolas which intersect tangentially at the origin: $\Gamma_1 \cap \Gamma_2 = \{(0,0)\}$. For each $j = 1,2$, the scalar symbol $\xi^2 + a_j x$ is microhyperbolic at $(0,0)$ in any direction (x^*,ξ^*) with $a_j x^* > 0$. The matrix-valued symbol $H(x,\xi)$ is microhyperbolic in any direction $x^* > 0$ [resp. $x^* < 0$] if both a_1, a_2 are positive [resp. negative], whereas $H(x,\xi)$ is not microhyperbolic in any direction if $a_1 a_2 < 0$.

The following theorem is due to Ivrii [22].

Theorem 2.1. *Let $u(x,h) \in L^2(\mathbb{R}^n;\mathbb{C}^N)$ satisfying $\|u\| \le 1$ be a solution to the system (2.1). Assume that $H_0(x,\xi)$ is microhyperbolic at a point $(x_0,\xi_0) \in \mathbb{R}^n \times \mathbb{R}^n$ in the direction $(x^*,\xi^*) \in (\mathbb{R}^n \times \mathbb{R}^n) \setminus (0,0)$. If there exists a neighborhood U of (x_0,ξ_0) such that*

$$\mathrm{WF}_h(\mathbf{u}) \cap \{(x,\xi) \in U; \xi^* \cdot (x-x_0) - x^* \cdot (\xi - \xi_0) < 0\} = \emptyset,$$

then $(x_0,\xi_0) \notin \mathrm{WF}_h(\mathbf{u})$.

Remark 2.6. *Let $g(x,\xi) := \xi^* \cdot x - x^* \cdot \xi$. Then the microhyperbolicity of H_0 at (x_0,ξ_0) in the direction $H_g(x_0,\xi_0)$ (Hamilton vector field of g) is expressed by*

$$\{H_0, gI_N\} + CH_0^2 \ge \frac{1}{C} \quad \text{at } (x_0,\xi_0),$$

and the conclusion of Theorem 2.1 is

$$\mathrm{WF}_h(\mathbf{u}) \cap \{(x,\xi) \in U; g(x,\xi) < g(x_0,\xi_0)\} = \emptyset \Rightarrow (x_0,\xi_0) \notin \mathrm{WF}_h(\mathbf{u}). \tag{2.3}$$

We see from Remark 2.6 that Theorem 2.1 implies that if there exists a local scalar escape function, then the semiclassical wave front set propagates in the direction where the escape function increases.

In particular in our models A, B and C, every point in the phase space is microhyperbolic. At the crossing point $v_+ = (0,\sqrt{E})$, $g(x,\xi) = x$ is an escape function, and hence $\mathscr{P}$ is microhyperbolic in the direction $(0,1)$, while at $v_- = (0,-\sqrt{E})$, $g(x,\xi) = -x$ is an escape function, i.e. $\mathscr{P}$ is microhyperbolic in the direction $(0,-1)$.

Remark 2.7. *We finally remark that the terminology microhyperbolic was introduced by Kawai and Kashiwara for a scalar analytic symbol. They say that an analytic symbol H is microhyperbolic at (x_0,ξ_0) in the direction H_g, if there exist a neighborhood U of (x_0,ξ_0) and a constant $\delta_0 > 0$ such that for all $(x,\xi) \in U$ and for all $0 < \delta < \delta_0$, one has*

$$H_0(x + i\delta\partial_\xi g(x_0,\xi_0), \xi - i\delta\partial_x g(x_0,\xi_0)) \neq 0. \tag{2.4}$$

This is weaker than the microhyperbolicity in the sense of Definition 2.5. In fact, the symbol $x\xi$ in dimension $n=1$ is not microhyperbolic in the sense of Definition 2.5 but microhyperbolic in the sense of (2.4) for any $g(x,\xi) := -\xi^* \cdot x + x^* \cdot \xi$ with $x^*\xi^* \neq 0$. The well known Kawai-Kashiwara theorem [26] states the propagation of analytic wave front set (2.3) under the microhyperbolicity (2.4).

2.2. **Landau-Zener model.** We take a simple example called *Landau-Zener model*:

$$H(x,\xi) = \begin{pmatrix} \xi - x & \epsilon \\ \epsilon & \xi + x \end{pmatrix}.$$

We will take $\epsilon = h$ later, but for the moment it is an independent small parameter. Its Weyl quantization is written in the form

$$H^W(x,hD) = \begin{pmatrix} A_- & \epsilon \\ \epsilon & A_+ \end{pmatrix} = hD + Q, \quad A_\pm = hD \pm x, \quad Q = \begin{pmatrix} -x & \epsilon \\ \epsilon & x \end{pmatrix}$$

If x is regarded as time variable, the equation

$$(hD + Q)u = 0 \tag{2.5}$$

is an evolution equation for a Hamiltonian Q. The parameter h is regarded as *adiabatic parameter*, namely it represents the slow time scale.

When $\epsilon = 0$, the eigenvalues $-x$ and x of the matrix Q cross at the origin $x = 0$. In other words, the characteristic sets $\{(x, \xi); \xi = x\}$ and $\{(x, \xi); \xi = -x\}$ cross transversally at the origin $(x, \xi) = (0, 0)$. The solution to (2.5) is written as

$$u = C_+ \begin{pmatrix} e^{\frac{i}{2h}x^2} \\ 0 \end{pmatrix} + C_- \begin{pmatrix} 0 \\ e^{-\frac{i}{2h}x^2} \end{pmatrix}.$$

Remark that the semiclassical wave front sets of the functions $e^{\pm\frac{i}{2h}x^2}$ are on $\{(x, \xi); \xi = \pm x\}$ respectively (see Proposition 2.2).

When $\epsilon > 0$, the eigenvalues of Q becomes $\pm\sqrt{x^2 + \epsilon^2}$ that avoid crossing with distance 2ϵ. Thus the two parameters h and ϵ play opposite roles. As h becomes small, the transition becomes small whereas as ϵ becomes small, the transition becomes large. In fact, Landau showed that the *transition probability* is given by $e^{-\frac{\pi\epsilon^2}{h}}$.

Let us consider an integral operator

$$(Tv)(x) := \int_{\mathbb{R}} e^{\frac{i}{h}\phi(x,y)} v(y) dy, \quad \phi(x, y) = \frac{x^2}{2} - \sqrt{2}xy + \frac{y^2}{2}.$$

The canonical transformation $(y, -\phi'_y(y)) \mapsto (x, \phi'_x(x))$ associated with the phase function $\phi(x, y)$ is the rotation with angle $-\pi/4$: $(y, \eta) \mapsto (x, \xi) = ((y + \eta)/\sqrt{2}, (-y + \eta)/\sqrt{2})$, which sends $\{(x, \xi); \xi = \pm x\}$ to $\{(x, \xi); \xi = 0\}$ and $\{(x, \xi); x = 0\}$ respectively.

We immediately see the following formulas.

$$A_+ Tv = \sqrt{2}\, T(hDv), \quad A_- Tv = -\sqrt{2}\, T(xv).$$

Put $u = Tv$. Then the system (2.5) becomes

$$\begin{pmatrix} -x & \frac{\epsilon}{\sqrt{2}} \\ \frac{\epsilon}{\sqrt{2}} & hD \end{pmatrix} v = 0. \tag{2.6}$$

Solving this reduced system, which means

$$\left(xhD + \frac{\epsilon^2}{2} \right) v_2 = 0, \quad v_1 = \frac{\epsilon}{2x} v_2,$$

we obtain the following distribution solutions

$$v_\pm(x, h) = \begin{pmatrix} \frac{\epsilon}{\sqrt{2}} Y(\pm x)|x|^{-i\frac{\epsilon^2}{2h}-1} \\ Y(\pm x)|x|^{-i\frac{\epsilon^2}{2h}} \end{pmatrix},$$

where $Y(t)$ is the Heaviside function.

Let us study the asymptotic behavior of the solution $u_+ = Tv_+$ to the equation (2.5):

$$u_+ = Tv_+ = \begin{pmatrix} \frac{\epsilon}{\sqrt{2}} I_\mu(x, h) \\ I_{\mu+1}(x, h) \end{pmatrix}, \quad \mu = -i\frac{\epsilon^2}{2h},$$

where $I_\mu(x, h)$ is the function defined by the following integral.

$$I_\mu(x, h) = \int_0^\infty e^{\frac{i}{h}\phi(x,y)} y^{\mu-1} dy.$$

This integral is well-defined for $\mu \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. It is important to remark here that if $0 < \epsilon = o(h^{\frac{1}{2}})$ as $h \to +0$, then $0 \neq \mu = o(1)$ and I_μ is well-defined.

Modulo $\mathcal{O}(h^\infty)$, there are two contributions to the asymptotic behavior of the integral I_μ. One is the endpoint $x = 0$ of the integral range, and the other is the critical point of the phase function $\phi(x, y)$.

First we compute the contribution from the endpoint. Since y is close to zero, we take the first two terms of the phase function to get

$$I_\mu^0(x,h) = e^{\frac{i}{2h}x^2} \int_0^\infty e^{\frac{i}{h}\sqrt{2}xy} y^{\mu-1} dy.$$

With the change of variable $\frac{i}{h}\sqrt{2}xy = -t$, we have for $x \neq 0$,

$$I_\mu^0(x,h) = e^{\frac{i}{2h}x^2} \int_0^\infty e^{-t} t^{\mu-1} dy \left(\frac{ih}{\sqrt{2}x}\right)^\mu = \Gamma(\mu) \left(\frac{ih}{\sqrt{2}x}\right)^\mu e^{\frac{i}{2h}x^2} \tag{2.7}$$

Second, we compute the contribution from the critical point. Since $\phi_y' = -\sqrt{2}x + y$, the critical point $y = \sqrt{2}x$ exists in the integration range $(0, \infty)$ only for $x > 0$. Fix $x > 0$, and take a cutoff function $\chi(y) \in C_0^\infty(\mathbb{R}_+)$ which is identically 1 near $y = x$ and identically 0 neear $y = 0$. We consider

$$I_\mu^c(x,h) = \int_0^\infty e^{\frac{i}{h}\phi(x,y)} y^{\mu-1} \chi(y) dy.$$

Since $\phi_{yy}'' = 1$, the stationary phase method leads to the asymptotic formula

$$I_\mu^c(x,h) \sim e^{i\frac{\pi}{4}} \sqrt{2\pi h} (\sqrt{2}x)^{\mu-1} e^{-\frac{i}{2h}x^2}. \tag{2.8}$$

Summing up, we have seen that the solution u_+ to the Landau-Zener system (2.5) has the asymptotic formula (2.7) for $x < 0$, while it is the sum of the two terms (2.7) and (2.8).

From the microlocal point of view, the above fact is interpreted as follows. Both (2.7) and (2.8) are of WKB form with phase functions $\frac{x^2}{2}$ and $-\frac{x^2}{2}$ respectively. Proposition 2.2 tells that their semiclassical wave front set is on the Lagrangian submanifold $\{(x, \xi); \xi = x\}$ and $\{(x, \xi); \xi = -x\}$ respectively. Thus we obtain

$$\mathrm{WF}_h(u_+) = \{(x, \xi); \xi = x\} \cup \{(x, \xi); \xi = -x, x > 0\}.$$

3. WKB SOLUTIONS

3.1. **Formal construction.** We formally construct a WKB solution $\mathbf{u} = {}^t(u_1, u_2)$ to the system

$$\mathscr{P}\mathbf{u} = E\mathbf{u}, \quad \mathbf{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{3.1}$$

in the WKB form

$$\mathbf{u}(x,h) = e^{i\phi(x)/h}\mathbf{a}(x,h), \quad \mathbf{a}(x,h) = \begin{pmatrix} a_1(x,h) \\ a_2(x,h) \end{pmatrix} \sim \sum_{k=0}^\infty h^k \begin{pmatrix} a_{1,k}(x) \\ a_{2,k}(x) \end{pmatrix}. \tag{3.2}$$

Since

$$e^{-\frac{i}{h}\phi(x)}(P_j - E)(e^{\frac{i}{h}\phi(x)} a_j) = \left((\phi')^2 + V_j - E\right) a_j + \frac{h}{i} L a_j - h^2 a_j'',$$

where L is the first order linear differential operator

$$L = 2\phi' \frac{d}{dx} + \phi'',$$

the equation (3.1) with (3.2) requires the coefficients $\mathbf{a}_k$ to satisfy

$$\begin{pmatrix} (\phi')^2 + V_1 - E & 0 \\ 0 & (\phi')^2 + V_2 - E \end{pmatrix} \mathbf{a}_k + \begin{pmatrix} -iL & W \\ W & -iL \end{pmatrix} \mathbf{a}_{k-1} + \frac{d^2}{dx^2}\mathbf{a}_{k-2} = 0, \tag{3.3}$$

for all $k = 0, 1, 2, \ldots$, with the convention that $\mathbf{a}_k \equiv 0$ for negative k. In particular, when $k = 0$, one has

$$\begin{pmatrix} (\phi')^2 + V_1 - E & 0 \\ 0 & (\phi')^2 + V_2 - E \end{pmatrix} \mathbf{a}_0 = 0. \tag{3.4}$$

In order to obtain a non-trivial sequence of coefficients, we should require either $(\phi')^2 + V_1 - E = 0$ or $(\phi')^2 + V_2 - E = 0$. Let ϕ satisfy the first one

$$(\phi')^2 + V_1(x) - E = 0. \tag{3.5}$$

This is the eikonal equation for the WKB solution to the single equation $(P_1 - E)u_1 = 0$. It determines the phase function ϕ_1 up to an additive constant. If x is in the classically allowed region $\{x; V_1(x) \leq E\}$, then the solution of (3.5) is written $\phi(x) = \pm\phi_1(x)$ where

$$\phi_j(x, x_0) = \int_{x_0}^{x} \sqrt{E - V_j(t)}dt, \quad j = 1, 2. \tag{3.6}$$

Here x_0 is an arbitrary point in the classically allowed region. We will usually take as this point a turning point.

With this $\phi = \phi_1$, the second condition in (3.4) becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & V_2 - V_1 \end{pmatrix} \mathbf{a}_0 = 0.$$

and hence we have

$$a_{2,0}(x) = 0, \tag{3.7}$$

unless $V_1(x) - V_2(x)$ vanishes identically.

The recurrence equation (3.3) becomes

$$L_1 a_{1,k-1} = i a''_{1,k-2} - iW a_{2,k-1}, \tag{3.8}$$

$$(V_2(x) - V_1(x))\, a_{2,k} = iL_1 a_{2,k-1} + a''_{2,k-2} - W a_{1,k-1}, \tag{3.9}$$

where $L{=}2\phi_1' \frac{d}{dx} + \phi_1''$.

The transport equation (3.8) with $k = 1$ gives

$$La_{1,0} = 0,$$

which is the first transport equation for the single equation $(P_1 - E)u_1 = 0$, and the solution is given up to a multiplicative constant by

$$a_{1,0}(x) = \frac{1}{(E - V_1(x))^{\frac{1}{4}}}.$$

Next, the transport equation (3.9) with $k = 1$ yields, with (3.7),

$$a_{2,1}(x) = \frac{W(x)}{(V_1(x) - V_2(x))(E - V_1(x))^{\frac{1}{4}}}$$

The second coefficient $a_{1,1}$ of the first entry satisfies

$$\begin{aligned} L_1 a_{1,1} &= i a_{1,0}'' - iW a_{2,1}, \\ &= \left(\frac{i}{(E - V_1(x))^{\frac{1}{4}}} \right)'' - i \frac{W(x)^2}{(V_1(x) - V_2(x))(E - V_1(x))^{\frac{1}{4}}}. \end{aligned}$$

This is a first order linear differential equation and is uniquely solved in any interval in the classically allowed region for p_1 free from turning point and crossing point when an initial condition $a_{1,1}(x_1) = 0$ is imposed at some point x_1 in this interval.

In the same way, we inductively determine $a_{1,k}$ by (3.8), and $a_{2,k}$ by (3.9) one after the other. Thus we obtain formal a power series solutions of the form

$$\begin{aligned} \mathbf{u}_1^{\pm}(x, h) &= e^{\pm \frac{i}{h}\phi_1(x)} \sum_{k=0}^{\infty} h^k \begin{pmatrix} a_{1,k}(x) \\ a_{2,k}(x) \end{pmatrix} \\ &= \frac{e^{\pm \frac{i}{h}\phi_1(x)}}{(E - V_1(x))^{\frac{1}{4}}} \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix} + h \begin{pmatrix} * \\ \frac{W(x)}{V_1(x) - V_2(x)} \end{pmatrix} + \cdots \right), \end{aligned} \tag{3.10}$$

where we omit the computation of $* = (E - V_1(x))^{\frac{1}{4}} a_{1,1}$.

Similarly, starting from the eikonal equation

$$(\phi_2')^2 + V_2(x) - E = 0, \tag{3.11}$$

we get other formal solutions

$$\mathbf{u}_2^{\pm}(x, h) = \frac{e^{\pm \frac{i}{h}\phi_2(x)}}{(E - V_2(x))^{\frac{1}{4}}} \left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + h \begin{pmatrix} \frac{W(x)}{V_2(x) - V_1(x)} \\ * \end{pmatrix} + \cdots \right). \tag{3.12}$$

As in the usual single Schrödinger equation case, these expressions of solutions have singularities at the turning points, i.e. the zeros of $E - V_j(x)$. In our matrix case, moreover, they have additional singularities at the zeros of $V_1(x) - V_2(x)$, that we will call *crossing points*.

These apparent singularities are due to the divergence of the infinite series. To give a sense to these divergent series, we take a Borel sum. Namely, we construct functions $\mathbf{u}_j^{\pm}(x, h)$ which have the infinite series as their asymptotic expansions as $h \to 0$. We denote them again by the same notations $\mathbf{u}_j^{\pm}(x, h)$. They are no longer solutions to our system (3.1), but are only quasi-modes, i.e. they satisfy

$$(\mathscr{P} - E)\mathbf{u}_j^{\pm} = \mathcal{O}(h^{\infty}), \tag{3.13}$$

uniformly in an interval contained in the classically allowed region away from turning points and crossing points. We see in the next section that they can be regarded as *microlocal solutions* when restricted to the characteristic set in the phase space.

We finally recall, before ending this section, that our construction of WKB solutions of the form (3.10) and (3.12) depends on the choice of the base point x_0 of the phase function and the base point x_1 of the symbol function. In the following we specify the choice of x_0 but we do not specify x_1. The change of x_1 results in an error of order h since this choice first appears for $a_{1,1}$ as we saw above.

3.2. **Microlocal WKB solutions.** Now let us observe the frequency set of the WKB solutions $\mathbf{u}_j^\pm$, $j = 1, 2$ that we constructed in the previous section. Let $\Gamma_t(E)$ be the set of turning points and $\Gamma_c(E)$ the set of microlocal crossing points:

$$\Gamma_t(E) = \Gamma_{1,t}(E) \cup \Gamma_{2,t}(E), \quad \Gamma_{j,t}(E) := \{(x, 0); V_j(x) = E\},$$
$$\Gamma_c(E) := \Gamma_1(E) \cap \Gamma_2(E) = \{(x, \xi); V_1(x) = V_2(x) \leq E,\ \xi = \pm\sqrt{E - V_1(x)}\}.$$

To each function $\mathbf{u}_j^\pm$, there corresponds a connected component γ of the set $\Gamma(E) \setminus (\Gamma_t(E) \cup \Gamma_c(E))$ in the phase space. This component is chosen in such a way that it belongs to $\Gamma_j^\pm(E) := \Gamma_j(E) \cap \{(x, \xi); \pm\xi > 0\}$ and $\mathbf{u}_j^\pm$ is defined in its x-projection $\pi(\gamma)$. According to this correspondence, we write

$$\mathbf{f}_\gamma := \mathbf{u}_j^\pm,$$

and call it *microlocal WKB solution* on γ. Recall that the WKB solution $\mathbf{u}_j^\pm$ depends on the choice of the base point $x_0 \in \pi(\gamma)$ for the phase function $\phi_j(x, x_0)$ (see (3.6)), and the base point x_1 of the symbol. When necessary, we will also denote $\mathbf{f}_\gamma = \mathbf{f}_{\gamma,\alpha}$ specifying the point α on γ such that $\pi(\alpha) = x_0$. The change of the choice of x_1 makes a change of $\mathcal{O}(h)$.

The microlocal WKB solution $\mathbf{f}_\gamma$ satisfies

$$(\mathscr{P} - E)\mathbf{f}_\gamma = \mathcal{O}(h^\infty) \quad \text{in } \pi(\gamma) \tag{3.14}$$

as we saw in the previous section. Moreover, we see from Proposition 2.3 that

$$\mathrm{WF}_h(\mathbf{f}_\gamma) \cap (\pi(\gamma) \times \mathbb{R}_\xi) = \gamma. \tag{3.15}$$

In fact, the set γ is a Lagrangian manifolds defined by $\xi = \pm\phi_j'(x) = \pm\sqrt{E - V_j(x)}$ for x in the classically allowed region for p_j.

On the contrary, to each connected component γ of the set $\Gamma(E) \setminus (\Gamma_t(E) \cup \Gamma_c(E))$, a microlocal WKB solution $\mathbf{f}_\gamma$ is associated. Moreover, the vector space of microlocal solutions on each γ is one-dimensional. In fact, if $\gamma \subset \Gamma_1(E)$, say, the operator $P_1 - E$ is reduced to the normal form hD_x while $P_2 - E$ is elliptic there. Therefore it is generated by $\mathbf{f}_\gamma$.

Summing up, we have the following proposition.

Proposition 3.1. *Let $\mathbf{u} \in L^2(\mathbb{R}; \mathbb{C}^2)$, $\|\mathbf{u}\|_{L^2} \leq 1$ be a solution to the system (3.1). Then for each connected component γ of the set $\Gamma(E) \setminus \Gamma_t(E) \cap \Gamma_c(E)$, there exists a complex number α_γ such that*

$$\mathbf{u} \equiv \alpha_\gamma \mathbf{f}_\gamma \quad \text{on } \gamma.$$

Remark 3.2. *The number α_γ is determined up to $\mathcal{O}(h^\infty)$ when the base points of the phase and the symbol are specified, but up to $\mathcal{O}(h)$ when only the base point of the phase is specified.*

4. Microlocal scattering matrix at a crossing point

If we know α_γ for all the connected component γ of $\Gamma(E) \setminus \Gamma_t(E) \cap \Gamma_c(E)$, we have all the microlocal information of $\mathbf{u}$ in the phase space since $\mathbf{u}$ is microlocally infinitely small outside the characteristic set $\Gamma(E)$ as stated in Proposition 2.3. This permits us to know the global behavior of $\mathbf{u}$ modulo $\mathcal{O}(h^\infty)$, which leads to the asymptotic study of eigenvalues or resonances. In fact, it holds that if a solution $\mathbf{u} \in L^2(\mathbb{R};\mathbb{C}^2)$, $\|\mathbf{u}\|_{L^2} \le 1$ to the system (3.1) is microlocally infinitely small everywhere in $I \times \mathbb{R}_\xi$ for an interval $I \subset \mathbb{R}_x$, then $\|\mathbf{u}\|_{L^2(I)} = \mathcal{O}(h^\infty)$. Namely, the microlocal behavior in $I \times \mathbb{R}_\xi$ determines the local behavior in I modulo $\mathcal{O}(h^\infty)$.

Thus the global study of solutions to the system (3.1) is reduced to the connection of the coefficients α_γ at each turning point and crossing point. The connection problem at a turning point is well known at least for the scalar case in general dimension as Maslov's theory. In the system case also, similar formulas hold as long as the turning point is not a crossing point.

4.1. **Connection at a turning point.** Let $(x^*, 0) \in \Gamma_t \setminus \Gamma_c$. Suppose for example it is a turning point of p_1: $(x^*, 0) \in \Gamma_{1,t}$. We assume it is a simple turning point, i.e.

$$V_1'(x^*) \neq 0.$$

Then, there are exactly two connected components of $\Gamma(E) \setminus (\Gamma_t(E) \cup \Gamma_c(E))$ which have the turning point $(x^*, 0)$ as an extremity. Let $\gamma^\flat$ be the one which has $(x^*, 0)$ as endpoint of the Hamiltonian flow H_{p_1} on it and $\gamma^\sharp$ the other one. In other words, $\gamma^\flat$ is the incoming classical trajectory to $(x^*, 0)$ and $\gamma^\sharp$ is the outgoing classical trajectory from $(x^*, 0)$.

Let $\mathbf{f}_\flat$ and $\mathbf{f}_\sharp$ be microlocal solutions defined on $\gamma_\flat$ and $\gamma_\sharp$ respectively, with phase functions $\phi_1(x, x^*)$ based on the turning point x^* (see (3.6)), and let $\alpha_\sharp = \alpha_{\gamma_\sharp}$ and $\alpha_\flat = \alpha_{\gamma_\flat}$ for short where $\alpha_{\gamma_\sharp}$ and $\alpha_{\gamma_\flat}$ are defined by Proposition 3.1. Under this setting, the following asymptotic connection formula holds. For the proof, we refer for example to [12].

Proposition 4.1. *Let $\mathbf{u} \in L^2(\mathbb{R};\mathbb{C}^2)$, $\|\mathbf{u}\|_{L^2} \le 1$ be a solution to the system (3.1). If*

$$\mathbf{u} \equiv \alpha_\flat \mathbf{f}_\flat \quad \text{on } \gamma_\flat,$$
$$\mathbf{u} \equiv \alpha_\sharp \mathbf{f}_\sharp \quad \text{on } \gamma_\sharp,$$

then it holds modulo $\mathcal{O}(h)$ that

$$\alpha_\sharp = -i\alpha_\flat.$$

4.2. **Connection at a crossing point.** Now we study the connection at a crossing point. Let $\rho := (x_0, \xi_0) \in \mathbb{R}_x \times \mathbb{R}_\xi$ be a microlocal crossing point $\rho \in \Gamma_c(\lambda) = \Gamma_1(\lambda) \cap \Gamma_2(\lambda)$. We assume without loss of generality that $x_0 = 0$ and

$$V_1(0) = V_2(0) = 0. \tag{4.1}$$

We moreover assume that $V_1(x)$ and $V_2(x)$ intersect at $x = 0$ at a finite order, i.e. there exists $n \in \mathbb{N}$ such that

$$V_1^{(k)}(0) - V_2^{(k)}(0) = 0 \quad (0 \le k \le m-1), \qquad V_1^{(m)}(0) - V_2^{(m)}(0) \neq 0. \tag{4.2}$$

We suppose $E_0 > 0$. Then for E close to E_0, one has $\xi_0 = \pm\sqrt{E}$ and there are two crossing points $\rho_+ := (0, \sqrt{E})$ and $\rho_- := (0, -\sqrt{E})$ symmetric with respect to the x-axis. The contact order at these points are both m. They are not turning points.

As in the previous case of a turning point, we denote $\gamma_{j,\flat}, \gamma_{j,\sharp}$ for each $j = 1, 2$ the incoming and the outgoing classical trajectories on $\Gamma_j(E)$ having ρ as their extremity. We write also $\alpha_{j,\flat}$, $\alpha_{j,\sharp}$ the coefficients $\alpha_{\gamma_{j,\flat}}$, $\alpha_{\gamma_{j_\sharp}}$ defined in Proposition 3.1.

Proposition 4.2. *Let $\mathbf{f}_{j,\flat}$, $\mathbf{f}_{j,\sharp}$ be the microlocal solutions to (3.1) defined on $\gamma_{j,\flat}, \gamma_{j,\sharp}$ for each $j = 1, 2$ with the phase base point at the microlocal crossing point ρ. Then there exists an h-dependent constant 2×2 matrix T such that if $\mathbf{u} \in L^2(\mathbb{R}; \mathbb{C}^2)$, $\|\mathbf{u}\|_{L^2} \leq 1$ is a solution to the system (3.1), and if $\alpha_{j,\flat}$, $\alpha_{j,\sharp}$ are defined for each $j = 1, 2$ by*

$$\mathbf{u} \equiv \alpha_{j,\flat}\mathbf{f}_{j,\flat} \quad \text{on } \gamma_{j,\flat},$$
$$\mathbf{u} \equiv \alpha_{j,\sharp}\mathbf{f}_{j,\sharp} \quad \text{on } \gamma_{j,\sharp},$$

then it holds modulo $\mathcal{O}(h)$ that

$$\begin{pmatrix}\alpha_{1,\sharp}\\ \alpha_{2,\sharp}\end{pmatrix} = T\begin{pmatrix}\alpha_{1,\flat}\\ \alpha_{2,\flat}\end{pmatrix}. \tag{4.3}$$

Proof. The vector space of microlocal solutions at the crossing point ρ is of dimension two and generated by the pair $(\mathbf{f}_{1,\flat}, \mathbf{f}_{2,\flat})$. In fact they are linearly independent and Theorem 2.1 implies that if $\mathbf{u} \equiv 0$ on $\gamma_{1,\flat} \cup \gamma_{2,\flat}$, then $\mathbf{u} \equiv 0$ on $\gamma_{1,\sharp} \cup \gamma_{2,\sharp}$ since the principal symbol of $\mathscr{P}$ is microhyperbolic at ρ_+ in the direction $(0, 1)$, and at ρ_- in the direction $(0, -1)$. □

Remark 4.3. *Here we do not specify the symbol base point, and hence the microlocal WKB solutions are defined up to $\mathcal{O}(h)$ only. That is why the microlocal scattering matrix is determined up to $\mathcal{O}(h)$.*

We call the matrix $T = (t_{jk})_{j,k=1,2}$ *microlocal scattering matrix*, since it describes the outgoing waves in terms of the incoming ones. The diagonal entries t_{jj} stands for the transmission of particles along $\Gamma_j(E)$ while the anti-diagonal entries t_{jk} stands for the 'change of trajectories' from $\Gamma_k(E)$ to $\Gamma_j(E)$ at the crossing point.

We are interested in the asymptotic behavior of the microlocal scattering matrix in the semiclassical limit $h \to 0$. For the sake of the application to the resonance asymptotics, we fix a real energy E_0 and consider the microlocal scattering matrix for E's in a complex neighborhood of E_0: $\mathcal{R}_{E_0}(\delta_1, \delta_2) := \{E \in \mathbb{C}; |\mathrm{Re}\, E - E_0| < \delta_1, |\mathrm{Im}\, E| < \delta_2\}$.

Let us begin with the generic case where the crossing point in question is not a turning point. The following theorem is due to [3].

Theorem 4.1.

$$T = \mathrm{Id} - ih^{\frac{1}{m+1}}\begin{pmatrix}0 & \omega\\ \overline{\omega} & 0\end{pmatrix} + \mathcal{O}(h^{\frac{1}{m+1}+\epsilon}) \quad \text{as } h \to 0, \tag{4.4}$$

where the constant $\omega \in \mathbb{C}$ is given by

$$\omega = \mu_m \left(\frac{2(m+1)!}{|V_1^{(m)}(0) - V_2^{(m)}(0)|}\right)^{\frac{1}{m+1}} \lambda_0^{-\frac{m}{2(m+1)}}\Gamma\left(\frac{m+2}{m+1}\right)W(0), \tag{4.5}$$

$$\mu_m = \begin{cases} \exp\left(\dfrac{i\pi}{2(m+1)}\,\mathrm{sgn}\left(V_2^{(m)}(0) - V_1^{(m)}(0)\right)\right) & \text{when } m \text{ is odd,}\\[2ex] \cos\left(\dfrac{\pi}{2(m+1)}\right) & \text{when } m \text{ is even.}\end{cases} \tag{4.6}$$

4.3. **Local exact solutions and microlocal scattering matrix.** Let $\rho = \rho_+ = (0, \sqrt{E_0})$ with $E_0 > 0$ be a microlocal crossing point in the phase space. In order to compute the microlocal scattering matrix T at ρ, we construct local exact solutions to the system (3.1) in a small neighborhood I of $x = 0$.

It is well known that there exist exact solutions $u_j^\pm$ to the scalar Schrödinger equations $(P_j - E)u = 0$ for each $j = 1, 2$ and E close to $E_0 > 0$ with the semiclassical asymptotic behavior in I:

$$u_j^\pm \sim (E - V_j(x))^{-\frac{1}{4}} \exp\left(\pm \frac{i}{h} \int_0^x \sqrt{E - V_j(t)} dt\right).$$

We can assume without loss of generality that the Wronskian of u_j^+ and u_j^- is equal to

$$\mathcal{W}(u_j^+, u_j^-) = u_j^+(u_j^-)' - u_j^-(u_j^+)' = -\frac{2i}{h}.$$

We suppose here that the smooth function $W(x)$ has a compact support in I. Let $K_{j,L}$ and $K_{j,R}$ be the operators acting on functions f in $C^\infty(\mathbb{R})$ defined by

$$K_{j,L} f := \frac{i}{2} \int_{-\infty}^x \left(u_j^+(x) u_j^-(y) - u_j^-(x) u_j^+(y)\right) W(y) f(y) dy, \tag{4.7}$$

$$K_{j,R} f := \frac{i}{2} \int_{\infty}^x \left(u_j^+(x) u_j^-(y) - u_j^-(x) u_j^+(y)\right) W(y) f(y) dy. \tag{4.8}$$

They satisfy

$$(P_j - E) K_{j,L} f = -hWf, \quad (P_j - E) K_{j,R} f = -hWf.$$

Using these operators, we construct exact solutions to the system (3.1) by a successive approximation. The following lemma can be proved using the same argument as the proof of the next Lemma 4.5.

Lemma 4.4. *For $S = L, R$, we have*

$$\|K_{1,S} K_{2,S}\|_{\mathcal{B}(C(I))} = \mathcal{O}(h^{\frac{1}{m+1}}), \quad \|K_{2,S} K_{1,S}\|_{\mathcal{B}(C(I))} = \mathcal{O}(h^{\frac{1}{m+1}}). \tag{4.9}$$

This lemma implies that the infinite sums

$$J_{1,S} := \sum_{k=0}^\infty (K_{1,S} K_{2,S})^k, \quad J_{2,S} := \sum_{k=0}^\infty (K_{2,S} K_{1,S})^k$$

converge for sufficiently small h in the space $\mathcal{B}(C(I))$ of bounded operators on $C(I)$.

For $S = L, R$, we set

$$\mathbf{w}_{1,S}^\pm := \begin{pmatrix} J_{1,S} u_1^\pm \\ K_{2,S} J_{1,S} u_1^\pm \end{pmatrix} = \begin{pmatrix} J_{1,S} u_1^\pm \\ J_{2,S} K_{2,S} u_1^\pm \end{pmatrix}, \tag{4.10}$$

$$\mathbf{w}_{2,S}^\pm := \begin{pmatrix} K_{1,S} J_{2,S} u_2^\pm \\ J_{2,S} u_2^\pm \end{pmatrix} = \begin{pmatrix} J_{1,S} K_{1,S} u_2^\pm \\ J_{2,S} u_2^\pm \end{pmatrix}. \tag{4.11}$$

These 8 functions $\mathbf{w}_{j,S}^\epsilon$ for $j = 1, 2$, $S = R, L$ and $\epsilon = +, -$ are all exact solutions to the system (3.1) in the interval I.

Let $\gamma_{j,\flat}^{\pm}$ be the incoming trajectories to the crossing point $\rho_{\pm}$ and $\gamma_{j,\sharp}^{\pm}$ the outgoing ones from $\rho_{\pm}$ along $\Gamma_j(\lambda_0)$ in $I \times \mathbb{R}_\xi$:

$$\gamma_{j,\flat}^{+} = \Gamma_j(\lambda_0) \cap \{(x,\xi) \in I \times \mathbb{R}; x < 0, \xi > 0\}, \quad \gamma_{j,\sharp}^{+} = \Gamma_j(\lambda_0) \cap \{(x,\xi) \in I \times \mathbb{R}; x > 0, \xi > 0\},$$
$$\gamma_{j,\flat}^{-} = \Gamma_j(\lambda_0) \cap \{(x,\xi) \in I \times \mathbb{R}; x > 0, \xi < 0\}, \quad \gamma_{j,\sharp}^{-} = \Gamma_j(\lambda_0) \cap \{(x,\xi) \in I \times \mathbb{R}; x < 0, \xi < 0\},$$

and let $\mathbf{f}_{j,\flat}^{\pm}$ and $\mathbf{f}_{j,\sharp}^{\pm}$ be the microlocal WKB solutions on $\gamma_{j,\flat}^{\pm}$ and $\gamma_{j,\sharp}^{\pm}$ respectively with phase base point at $\rho_{\pm}$.

We easily see the asymptotic behavior of $\mathbf{w}_{j,L}^{\pm}$ on the left of the origin $I \cap \{x < 0\}$ and that of $\mathbf{w}_{j,R}^{\pm}$ on the right of the origin $I \cap \{x > 0\}$, and we deduce their microlocal behavior modulo $\mathcal{O}(h)$ on each trajectory as follows.

Lemma 4.5. *The exact solutions $\mathbf{w}_{j,S}^{\pm}$, $j = 1, 2$, $S = L, R$ microlocally behave like*

$$\mathbf{w}_{j,L}^{+} \equiv \begin{cases} \mathbf{f}_{j,\flat}^{+} & \text{on } \gamma_{j,\flat}^{+}, \\ \mathbf{0} & \text{on } \gamma_{j,\sharp}^{-} \cup \gamma_{\hat{j},\flat}^{+} \cup \gamma_{\hat{j},\sharp}^{-}, \end{cases} \qquad \mathbf{w}_{j,R}^{+} \equiv \begin{cases} \mathbf{f}_{j,\sharp}^{+} & \text{on } \gamma_{j,\sharp}^{+}, \\ \mathbf{0} & \text{on } \gamma_{j,\flat}^{-} \cup \gamma_{\hat{j},\sharp}^{+} \cup \gamma_{\hat{j},\flat}^{-}, \end{cases} \tag{4.12}$$

$$\mathbf{w}_{j,L}^{-} \equiv \begin{cases} \mathbf{f}_{j,\sharp}^{-} & \text{on } \gamma_{j,\sharp}^{-}, \\ \mathbf{0} & \text{on } \gamma_{j,\flat}^{+} \cup \gamma_{\hat{j},\sharp}^{-} \cup \gamma_{\hat{j},\flat}^{+}, \end{cases} \qquad \mathbf{w}_{j,R}^{-} \equiv \begin{cases} \mathbf{f}_{j,\flat}^{-} & \text{on } \gamma_{j,\flat}^{-}, \\ \mathbf{0} & \text{on } \gamma_{j,\sharp}^{+} \cup \gamma_{\hat{j},\flat}^{-} \cup \gamma_{\hat{j},\sharp}^{+}, \end{cases} \tag{4.13}$$

modulo $\mathcal{O}(h)$ as $h \to 0$.

Proof. We only prove for $\mathbf{w}_{1,L}^{+}$. Recall that

$$\mathbf{w}_{1,L}^{+} = \begin{pmatrix} J_{1,L} u_1^{+} \\ J_{2,L} K_{2,L} u_1^{+} \end{pmatrix},$$

and

$$\begin{aligned} K_{2,L} u_1^{+}(x) &= \frac{i}{2} \int_{-\infty}^{x} \left(u_2^{+}(x) u_2^{-}(y) - u_2^{-}(x) u_2^{+}(y) \right) W(y) u_1^{+}(y) dy \\ &= \frac{i}{2} u_2^{+}(x) \int_{-\infty}^{x} u_2^{-}(y) u_1^{+}(y) W(y) dy - \frac{i}{2} u_2^{-}(x) \int_{-\infty}^{x} u_2^{+}(y) u_1^{+}(y) W(y) dy. \end{aligned}$$

The last two integrals are oscillatory integrals with integrand

$$u_2^{-}(y) u_1^{+}(y) W(y) = \frac{W(y)}{(E - V_1(y))^{\frac{1}{4}} (E - V_2(y))^{\frac{1}{4}}} \exp\left(\frac{i}{h} \int_0^y (E - V_1(t))^{\frac{1}{2}} - (E - V_2(t))^{\frac{1}{2}} dt \right),$$

$$u_2^{+}(y) u_1^{+}(y) W(y) = \frac{W(y)}{(E - V_1(y))^{\frac{1}{4}} (E - V_2(y))^{\frac{1}{4}}} \exp\left(\frac{i}{h} \int_0^y (E - V_1(t))^{\frac{1}{2}} + (E - V_2(t))^{\frac{1}{2}} dt \right).$$

The first integrand has a critical point at the crossing point $y = 0$ but it is outside the integration range when $x < 0$, and the second one has no critical point. Therefore, we see by an integration by parts that the integrals are both $\mathcal{O}(h)$. The only term which is not of order h is then u_1^{+} in the first entry.

On the other hand, the microlocal solution $\mathbf{f}_{1,\flat}^{+}$ also behaves like ${}^t(u_1^{+}, 0)$ modulo $\mathcal{O}(h)$. This ends the proof. □

The 4 solutions $(\mathbf{w}_{1,L}^+, \mathbf{w}_{2,L}^+, \mathbf{w}_{1,L}^-, \mathbf{w}_{2,L}^-)$ as well as the 4 solutions$(\mathbf{w}_{1,R}^+, \mathbf{w}_{2,R}^+, \mathbf{w}_{1,R}^-, \mathbf{w}_{2,R}^-)$ make a basis of exact solutions to the system $(\mathscr{P} - E)\mathbf{u} = \mathbf{0}$ in I. Therefore there exists a constant 4×4 matrix A such that

$$(\mathbf{w}_{1,L}^+, \mathbf{w}_{2,L}^+, \mathbf{w}_{1,L}^-, \mathbf{w}_{2,L}^-) = (\mathbf{w}_{1,R}^+, \mathbf{w}_{2,R}^+, \mathbf{w}_{1,R}^-, \mathbf{w}_{2,R}^-)A. \tag{4.14}$$

According to the previous lemma, we find the following lemma:

Lemma 4.6. *The microlocal scattering matrix T at the crossing point $\rho_+ = (0, \sqrt{E_0})$ is equal modulo $\mathcal{O}(h)$ to the 2×2 block matrix A_{11} of*

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

defined by (4.14).

We are now going to compute A at the principal level modulo $\mathcal{O}(h^{\frac{2}{m+1}})$ in the semiclassical limit. We rewrite the definition (4.14) of A in the form

$$\begin{pmatrix} \mathbf{w}_{1,L}^+ & \mathbf{w}_{2,L}^+ & \mathbf{w}_{1,L}^- & \mathbf{w}_{2,L}^- \\ (\mathbf{w}_{1,L}^+)' & (\mathbf{w}_{2,L}^+)' & (\mathbf{w}_{1,L}^-)' & (\mathbf{w}_{2,L}^-)' \end{pmatrix} = \begin{pmatrix} \mathbf{w}_{1,R}^+ & \mathbf{w}_{2,R}^+ & \mathbf{w}_{1,R}^- & \mathbf{w}_{2,R}^- \\ (\mathbf{w}_{1,R}^+)' & (\mathbf{w}_{2,R}^+)' & (\mathbf{w}_{1,R}^-)' & (\mathbf{w}_{2,R}^-)' \end{pmatrix} A, \tag{4.15}$$

and look at this identity at $x = 0$. Here $'$ stands for the derivative with respect to x. Notice that the both sides are 4×4 matrices. The first column vector on the left hand side is

$$^t(u_1^+(0), (K_{2,L}u_1^+)(0), (u_1^+)'(0), (K_{2,L}u_1^+)'(0)).$$

The term $(\ K_{2,L}u_1^+)(0)$ is expressed in a linear combination of $u_2^+(0)$ and $u_2^-(0)$.

$$\begin{aligned}
(\ K_{2,L}u_1^+)(0) &= \frac{i}{2}\int_{-\infty}^{x} \left(u_2^+(x)u_2^-(y) - u_2^-(x)u_2^+(y)\right) W(y)u_1^+(y)dy|_{x=0} \\
&= \frac{i}{2}\int_{-\infty}^{0} \left(u_2^+(0)u_2^-(y) - u_2^-(0)u_2^+(y)\right) W(y)u_1^+(y)dy \\
&= c_{1L}^+ u_2^+(0) + c_{1L}^- u_2^-(0),
\end{aligned}$$

with

$$c_{1L}^+ = \frac{i}{2}\int_{-\infty}^{0} u_2^-(y)u_1^+(y)W(y)dy, \quad c_{1L}^- = -\frac{i}{2}\int_{-\infty}^{0} u_2^+(y)u_1^+(y)W(y)dy.$$

Similarly, $(K_{2,L}u_1^+)'(0)$ is written in linear combination of $(u_2^+)'(0)$ and $(u_2^-)'(0)$:

$$(K_{2,L}u_1^+)'(0) = c_{1L}^+(u_2^+)'(0) + c_{1L}^-(u_2^-)'(0).$$

We define a matrix B by

$$B = \begin{pmatrix} u_1^+(0) & 0 & u_1^-(0) & 0 \\ 0 & u_2^+(0) & 0 & u_2^-(0) \\ (u_1^+)'(0) & 0 & (u_1^-)'(0) & 0 \\ 0 & (u_2^+)'(0) & 0 & (u_2^-)'(0) \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} \mathbf{w}_{1,L}^+ & \mathbf{w}_{2,L}^+ & \mathbf{w}_{1,L}^- & \mathbf{w}_{2,L}^- \\ (\mathbf{w}_{1,L}^+)' & (\mathbf{w}_{2,L}^+)' & (\mathbf{w}_{1,L}^-)' & (\mathbf{w}_{2,L}^-)' \end{pmatrix}|_{x=0} = B(I + C_L),$$

$$\begin{pmatrix} \mathbf{w}_{1,R}^+ & \mathbf{w}_{2,R}^+ & \mathbf{w}_{1,R}^- & \mathbf{w}_{2,R}^- \\ (\mathbf{w}_{1,R}^+)' & (\mathbf{w}_{2,R}^+)' & (\mathbf{w}_{1,R}^-)' & (\mathbf{w}_{2,R}^-)' \end{pmatrix}|_{x=0} = B(I + C_R),$$

where C_S, $S = L, R$ are 4×4 matrices of the form

$$C_S = \begin{pmatrix} 0 & c_{2S}^+ & 0 & c_{4S}^+ \\ c_{1S}^+ & 0 & c_{3S}^+ & 0 \\ 0 & c_{2S}^- & 0 & c_{4S}^- \\ c_{1S}^- & 0 & c_{3S}^- & 0 \end{pmatrix}.$$

The entries $c_{jL}^\pm$ are given by

$$c_{2L}^+ = \frac{i}{2}\int_{-\infty}^0 u_1^-(y)u_2^+(y)W(y)dy, \qquad c_{2L}^- = -\frac{i}{2}\int_{-\infty}^0 u_1^+(y)u_2^+(y)W(y)dy,$$
$$c_{3L}^+ = \frac{i}{2}\int_{-\infty}^0 u_2^-(y)u_1^-(y)W(y)dy, \qquad c_{3L}^- = -\frac{i}{2}\int_{-\infty}^0 u_2^+(y)u_1^-(y)W(y)dy,$$
$$c_{4L}^+ = \frac{i}{2}\int_{-\infty}^0 u_1^-(y)u_2^-(y)W(y)dy, \qquad c_{4L}^- = -\frac{i}{2}\int_{-\infty}^0 u_1^+(y)u_2^-(y)W(y)dy,$$

and $c_{jR}^\pm$ are given similarly with the lower endpoint of the integral replaced by $+\infty$.

Remark here that C_L, C_R are of $\mathcal{O}(h^{\frac{1}{m+1}})$. In fact, as in the proof of Lemma 4.5, the entries $c_{jS}^\pm$ are oscillatory integrals possibly with a critical point at the origin. The order of this critical point is the contact order $n = m$ of the two potentials, and hence by the degenerate stationary phase method, we see that it is of order $h^{\frac{1}{m+1}}$.

Then we obtain, for h small enough,

$$\begin{aligned} A &= (I + C_R)^{-1}(I + C_L) \\ &= I + C_L - C_R + \mathcal{O}(h^{\frac{2}{m+1}}) \\ &= I + \begin{pmatrix} 0 & c_2^+ & 0 & c_4^+ \\ c_1^+ & 0 & c_3^+ & 0 \\ 0 & c_2^- & 0 & c_4^- \\ c_1^- & 0 & c_3^- & 0 \end{pmatrix} + \mathcal{O}(h^{\frac{2}{m+1}}), \end{aligned}$$

where

$$c_1^+ = \frac{i}{2}\int_{-\infty}^\infty u_1^+(y)u_2^-(y)W(y)dy, \qquad c_1^- = -\frac{i}{2}\int_{-\infty}^\infty u_1^+(y)u_2^+(y)W(y)dy,$$
$$c_2^+ = \frac{i}{2}\int_{-\infty}^\infty u_1^-(y)u_2^+(y)W(y)dy, \qquad c_2^- = -\frac{i}{2}\int_{-\infty}^\infty u_1^+(y)u_2^+(y)W(y)dy,$$
$$c_3^+ = \frac{i}{2}\int_{-\infty}^\infty u_1^-(y)u_2^-(y)W(y)dy, \qquad c_3^- = -\frac{i}{2}\int_{-\infty}^\infty u_1^-(y)u_2^+(y)W(y)dy,$$
$$c_4^+ = \frac{i}{2}\int_{-\infty}^\infty u_1^-(y)u_2^-(y)W(y)dy, \qquad c_4^- = -\frac{i}{2}\int_{-\infty}^\infty u_1^+(y)u_2^-(y)W(y)dy.$$

The semiclassical asymptotics of these entries are obtained by the stationary phase method again as in the proof of of Lemma 4.5. The off-diagonal entries c_1^+, c_2^+ of the block A_{11}, which is the microlocal scattering matrix T at the microlocal crossing point $\rho_+ = (0, \sqrt{\lambda})$, and c_3^-, c_4^- of the block A_{22} have a degenerate stationary point at the origin, whereas those in the blocks A_{12}, A_{21} have no critical point.

More precisely, we have the following asymptotic formula, which gives Theorem 4.1:

$$c_1^+ = -i\omega h^{\frac{1}{m+1}} + \mathcal{O}(h^{\frac{2}{m+1}}), \quad c_2^+ = -i\overline{\omega} h^{\frac{1}{m+1}} + \mathcal{O}(h^{\frac{2}{m+1}}), \tag{4.16}$$

where ω is given by (4.5).

References

[1] S. Ashida : *Molecular predissociation resonances below an energy level crossing.* Asymptot. Analysis, vol. 107, no 3-4 (2018), pp. 135-167.

[2] M. Assal, S. Fujiié : *Eigenvalue splitting of polynomial order for a system of Schrödinger operators with energy-level crossing.* Comm. Math. Physics 386 (2021), pp. 1519-1550.

[3] M. Assal, S. Fujiié, K. Higuchi : *Semiclassical resonance asymptotics for systems with degenerate crossings of classical trajectories.* International Mathematics Research Notices, (2023) rnad290, http://doi.org/10.1093/imrn/rnad290.

[4] M. Assal, S. Fujiié, K. Higuchi : *Transition of the semiclassical resonance width across a tangentially crossing energy-level.* J. Math. Pure Appl., 191:103634, (2023).

[5] H. Baklouti : *Asymptotique des largeurs de résonances pour un modèle d'effet tunnel microlocal.* Ann. Inst. H. Poincare Phys. Théor., 68 (1998); no. 2, 179-228.

[6] J.-F. Bony, S. Fujiié, T. Ramond, M. Zerzeri : *Resonances for Homoclinic Trapped Sets.* Astérisque Bofuraze, Société Mathématique de France, 2018.

[7] Y. Colin de Verdière : *The level crossing problem in semi-classical analysis I. The symmetric case.* Ann. Institut Fourier 53, no. 4 (2003), pp. 1023-1054.

[8] Y. Colin de Verdière : *The level crossing problem in semi-classical analysis. II. The Hermitian case.* Ann. Institut Fourier 359, no. 5 (2004), pp. 1423-1441.

[9] M. Dimassi, S. Sjöstrand : *Spectral Asymptotics in the Semi-Classical Limit.* Cambridge University Press, 1999.

[10] S. Fujiié, A. Martinez, T. Watanabe : *Widths of resonances at an energy-level crossing I: Elliptic interaction.* J. Diff. Eq. 260 (2016), pp. 4051-4085.

[11] S. Fujiié, A. Martinez, T. Watanabe : *Widths of resonances at an energy-level crossing II: Vector field interaction.* J. Diff. Eq. 262 (2017), pp. 5880-5895.

[12] S. Fujiié, A. Martinez, T. Watanabe : *Widths of resonances above an energy-level crossing.* J. Funct. Anal. 280, no. 6 (2021), 108918.

[13] A. Grigis, A. Martinez : *Resonance widths in a case of multidimensional phase space tunneling.* Asymptot. Anal., 91 (2015), no. 1, 33-904.

[14] A. Grigis, A. Martinez, : *Resonance widths for the molecular predissociation.* Anal. PDE, 7 (2014), no. 5, 1027-1055.

[15] B. Helffer, J. Sjöstrand : *Résonances en limite semiclassique.* Bull. Soc. Math. France, Mémoire No. 24-25, 1986.

[16] B. Helffer, J. Sjöstrand : *Semiclassical analysis for Harper's equation. III. Cantor structure of the spectrum.* Mém. Soc. Math. France, (1989), no.39 1-124.

[17] K. Higuchi : *Resonances free domain for systems of Schrödinger operators above an energy-level crossing.* Rev. Math. Phys. **33** (2021), no.3 article no. 2150007.

[18] K. Higuchi : *Résonances semiclassiques engendrées par des croisements de trajectoires classiques*, C. R . Acad. Sci. 359, issue 6 (2021), pp. 657-663..

[19] K. Higuchi, V. Louatron, K. Taira : *A microlocal Cauchy problem through a crossing point of Hamiltonian flows*, ..

[20] L. Hörmander : *The analysis of linear partial differential operators I. Distribution theory and Fourier analysis.* Springer-Verlag, Berlin 1983.

[21] A. Ifa, H. Louati, M. Rouleux : *Bohr-Sommerfeld quantization rules revisited: The method of positive commutators.* J. Math. Sci. Univ. Tokyo, 25 (2018), 91-127.

[22] V. Ivrii : *Microlocal Analysis, Sharp Spectral Asymptotics and Appliciations I V.* Springer-Verlag, Berlin 2019.

[23] T. Kawai, M. Kashiwara : *Microhyperbolic Pseudodifferential Operators I.* J. Math. Soc. Japan, 27 (1975), 359-404.

[24] M. Klein, A. Martinez, R. Seiler, X. W. Wang : *On the Born-Oppenheimer expansion for polyatomic molecules.* Comm. Math. Phys. 143(3), 607-639 (1992).
[25] V. V. Kucherenko : *Asymptotics of the solution of the system* $A(x, -ih\partial/\partial x)u = 0$ *as* $h \to 0$. Izv. Akad. Nauk SSSR Ser. Mat., vol. 8, no. 3 (1974), 631-666.
[26] A. V. Krivko, V. V. Kucherenko : *Semiclassical Asymptotics of the Matrix Sturm-Liouville Problem.* Mathematical Notes, vol. 80, no. 1 (2006), 136-140.
[27] V. Louatron : *Semiclassical resonances for matrix Schrödinger operators with vanishing interactions at crossings of classical trajectories.* arXiv:2306.02350 .
[28] A. Martinez : *Estimates on complex interactions in phase space.* Math. Nachr. 167 (1994), 203-254.
[29] A. Martinez : *An Introduction to Semiclassical and Microlocal Analysis.* Springer-Verlag New-York, UTX Series, 2002.
[30] A. Martinez : *Resonance free domains for non globally analytic potentials.* Ann. Henri Poincaré 3 (2002), No. 4, 739-756.
[31] S. Nakamura : *On an example of phase-space tunneling.* Ann. Inst. H. Poincare Phys. Theor., 63 (1995), no. 2, 211-229.
[32] F. W. J. Olver, : *Asymptotics and Special Functions.* Academic Press, New York, 1974.
[33] J. Sjöstrand : *Density of states oscillations for magnetic Schrödinger operators.* Mathematics in Science and Engineering, Volume 186, 1992, Pages 295-345.
[34] S. Wakabayashi : *Singularities of solutions of the Cauchy problem for symmetric hyperbolic systems,* Commun. Part. Diff. Eq., 9 (12): 1147–1177 (1984).
[35] D.R. Yafaev : *The semiclassical limit of eigenfunctions of the Schrödinger equation and the Bohr-Sommerfeld quantization condition, revisited.* Algebra i Analiz, 22 (2010), no. 6, 270-291; translation in St. Petersburg Math. J., 22 (2011), no. 6, 1051-1067.
[36] M. Zworski : *Semiclassical Analysis, Graduate Studies in Mathematics, 138.* American Mathematical Soc., 2012.

Setsuro Fujiie, Department of Mathematical Sciences, Ritsumeikan University, 1-1-1 Noji-Higashi, Kusatsu, 525-8577, Japan. e-mail: fujiie@fc.ritsumei.ac.jp